Definition 1. *The Riemann Zeta function is defined as*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for $Re(s) > 1$.

Theorem 1.

$$\zeta(s) = -s \int_0^{\infty} \frac{x - [x] - \frac{1}{2}}{x^{s+1}} dx \tag{2}$$

for $-1 < Re(s) < 0$.

The following is a more detailed version of the proof found in [1].

Proof. Consider the equation:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \int_1^{\infty} \frac{x - [x] - \frac{1}{2}}{x^{s+1}} dx = s \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{s-1} + \frac{1}{2} \tag{3}$$

this is a corollary of the Representation by Euler-Maclaurin Formula, a detailed proof can be found on our site.

Remember that since $[x] - x + \frac{1}{2}$ is bounded, this integral converges for $Re(s) > 0$ and converges uniformly in any finite region to the right of $Re(s) = 0$, this is therefore an analytic continuation of $\zeta(s)$ up to $Re(s) = 0$.

Actually, let's observe now that 3 gives the analytic continuation of $\zeta(s)$ for $Re(s) > -1$, to do this we will check that the right hand side integral converges when $Re(s) > -1$:

Calling

$$f(x) := [x] - x + \frac{1}{2}, \qquad \text{and} \qquad f_1(x) := \int_1^x f(y)dy.$$

we find that both functions are bounded, since by definition $f(x) \leq \frac{3}{2}$ and $f_1(x)$ can be estimated using the fact that, for any integer k:

$$\begin{aligned}\int_k^{k+1} f(y)dy &= \int_k^{k+1} k - y + \frac{1}{2}dy = k\int_k^{k+1} 1dy - \int_k^{k+1} ydy + \frac{1}{2}\int_k^{k+1} 1dy \\ &= k - \frac{(k+1)^2}{2} + \frac{k^2}{2} + \frac{1}{2} = \frac{2k - k^2 - 2k - 1 + k^2 + 1}{2} = 0\end{aligned} \tag{4}$$

where we used the fact that in the interval $[k, k+1]$ we have $[y] = k$.

This implies that:

$$f_1(x) = \int_1^x f(y)dy = \int_{[x]}^x f(y)dy = \int_{[x]}^x [y]-y+\frac{1}{2}dy \leq \int_{[x]}^x 1+\frac{1}{2}dy \leq \frac{3}{2}(x-[x]) \leq \frac{3}{2}$$

and $f_1(k) = 0$ for any integer $k \geq 1$.

Hence, integrating by parts:

$$\begin{aligned}\int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}}dx &= \left[\frac{f_1(x)}{x^{s+1}}\right]_1^\infty + (s+1)\int_1^\infty \frac{f_1(x)}{x^{s+2}}dx = \lim_{x\to\infty}\left(\frac{f_1(x)}{x^{s+1}}\right) - f_1(0) + (s+1)\int_1^\infty \frac{f_1(x)}{x^{s+2}}dx \\ &= (s+1)\int_1^\infty \frac{f_1(x)}{x^{s+2}}dx \leq \frac{3}{2}(s+1)\int_1^\infty \frac{1}{x^{s+2}}dx \leq \frac{3}{2}(s+1)\frac{1}{s+1}\lim_{x\to\infty}\left(1 - \frac{1}{x^{s+1}}\right) \leq \frac{3}{2}\end{aligned} \tag{5}$$

where we used that $f_1(1) = 0$, $f_1(x) \leq \frac{3}{2}$ and that $Re(s) > -1$; this proves that the integral in 3 converges for $Re(s) > -1$.

Also, when $Re(s) < 0$:

$$\begin{aligned}s\int_0^1 \frac{[x] - x + \frac{1}{2}}{x^{s+1}}dx &= s\int_0^1 \frac{-x + \frac{1}{2}}{x^{s+1}}dx = -s\int_0^1 \frac{1}{x^s}dx + s\int_0^1 \frac{1}{2x^{s+1}}dx \\ &= -s\int_0^1 \frac{1}{x^s}dx - \frac{1}{2} = \frac{s}{s-1} - \frac{1}{2} = \frac{s}{s-1} - \frac{1}{2} + 1 - 1 = \frac{s}{s-1} - 1 + \frac{1}{2} \\ &= \frac{s}{s-1} - \frac{s-1}{s-1} + \frac{1}{2} = \frac{1}{s-1} + \frac{1}{2}.\end{aligned} \tag{6}$$

Hence, for $-1 < Re(s) < 0$, where both 3 and 6 are valid:

$$\begin{aligned}&\zeta(s) = s\int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}}dx + \frac{1}{s-1} + \frac{1}{2} = \\ &s\int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}}dx + s\int_0^1 \frac{[x] - x + \frac{1}{2}}{x^{s+1}}dx = s\int_0^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}}dx.\end{aligned} \tag{7}$$

□

References

[1] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.