Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}} \tag{2}$$

where γ *is the* ***Euler-Mascheroni constant***, $\gamma := \lim_{m\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{m} - \log m\right)$.

Proof. We will prove that:

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \frac{e^{\frac{s}{n}}}{1 + \frac{s}{n}} \tag{3}$$

which obviously implies 2.

Using the definition of the constant γ, write the right side of equation 3 as:

$$\begin{aligned}
\frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \frac{e^{\frac{s}{n}}}{1 + \frac{s}{n}} &= \lim_{m\to\infty} \frac{e^{-s-\frac{s}{2}-\cdots+s\log m}}{s} \prod_{n=1}^{m} \frac{e^{\frac{s}{n}}}{1 + \frac{s}{n}} \\
&= \lim_{m\to\infty} \frac{e^{-s-\frac{s}{2}-\cdots+s\log m} e^{s+\frac{s}{2}+\cdots+\frac{s}{m}}}{s} \prod_{n=1}^{m} \frac{1}{1 + \frac{s}{n}} \\
&= \lim_{m\to\infty} \frac{e^{s\log m}}{s} \prod_{n=1}^{m} \frac{1}{1 + \frac{s}{n}} = \lim_{m\to\infty} \frac{m^s}{s} \prod_{n=1}^{m} \frac{1}{1 + \frac{s}{n}}.
\end{aligned} \tag{4}$$

Notice now that, by definition of the factorial:

$$m = \frac{m!}{(m-1)!} = \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{m}{m-1} = \prod_{n=1}^{m-1} \left(1 + \frac{1}{n}\right) = \left(1 + \frac{1}{m}\right)^{-1} \prod_{n=1}^{m} \left(1 + \frac{1}{n}\right).$$

Substitute this expression of m in equation 4:

$$\begin{aligned}
\frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \frac{e^{\frac{s}{n}}}{1+\frac{s}{n}} &= \lim_{m\to\infty} \frac{1}{s}\left(1+\frac{1}{m}\right)^{-s} \prod_{n=1}^{m}\left(1+\frac{1}{n}\right)^{s} \prod_{n=1}^{m} \frac{1}{1+\frac{s}{n}} \\
&= \lim_{m\to\infty} \frac{1}{s}\left(1+\frac{1}{m}\right)^{-s} \prod_{n=1}^{m}\left(1+\frac{1}{n}\right)^{s} \frac{1}{1+\frac{s}{n}} \\
&= \frac{1}{s} \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1} = \Gamma(s).
\end{aligned} \tag{5}$$

The last product is exactly Euler's Infinite Product formula for the Gamma Function. □