Theorem 1.

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1}$$

where the symbol "$\approx$" denotes asymptotically equal.
*This is known as **Stirling's Formula.***

Proof. Remember the relation between the factorial and the Gamma Function:

$$\Gamma(n+1) = \int_0^\infty e^{-t}t^n dt = n!$$

for $n \geq 0$. Change variable to $x = \frac{(t-n)}{\sqrt{n}}$ to obtain:

$$\begin{aligned} n! &= \int_0^\infty e^{-t}t^n dt = \int_{-\sqrt{n}}^\infty e^{-(\sqrt{n}x+n)}(\sqrt{n}x+n)^n\sqrt{n}dx \\ &= \frac{n^n\sqrt{n}}{e^n}\int_{-\sqrt{n}}^\infty e^{-\sqrt{n}x}\left(\frac{x}{\sqrt{n}}+1\right)^n dx = \sqrt{n}\left(\frac{n}{e}\right)^n\int_{-\sqrt{n}}^\infty e^{-\sqrt{n}x}\left(\frac{x}{\sqrt{n}}+1\right)^n dx. \end{aligned} \tag{2}$$

Notice that the terms outside the integral are exactly those that appear in the final formula. Therefore we are left with proving that:

$$\int_{-\sqrt{n}}^\infty e^{-\sqrt{n}x}\left(\frac{x}{\sqrt{n}}+1\right)^n dx \to \sqrt{2\pi}.$$

We will prove that:

$$\lim_{n\to\infty}\int_{-\sqrt{n}}^\infty e^{-\sqrt{n}x}\left(\frac{x}{\sqrt{n}}+1\right)^n = \int_{-\infty}^\infty e^{-\frac{x^2}{2}} = \sqrt{2\pi}.$$

Extend the function to the whole real axis by defining $f_n(x)$ as:

$$f_n(x) := \begin{cases} 0 & \text{if } x \leq -\sqrt{n}, \\ e^{-\sqrt{n}x}\left(1+\frac{x}{\sqrt{n}}\right)^n & \text{if } x \geq -\sqrt{n}, \end{cases}$$

so that

$$n! = \sqrt{n}\left(\frac{n}{e}\right)^n \int_{-\infty}^{\infty} f_n(x)dx.$$

Fix $x \in \mathbb{R}$. Knowing that we have to compute the limit as $n \to \infty$ and x is a fixed value, we can always suppose n much bigger than $|x|$. In this case, we only need the second part of the definition of $f_n(x)$, that is to say:

$$f_n(x) = e^{-\sqrt{n}x}\left(1 + \frac{x}{\sqrt{n}}\right)^n$$

$$\Downarrow$$

$$\log(f_n(x)) = n\log\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x.$$

For $n > 4x^2$ we have $\left|\frac{x}{\sqrt{n}}\right| < \frac{1}{2}$. Hence we can use the Taylor expansion of the logarithmic function, $\log(1+x) = x - \frac{x^2}{2} + \mathcal{O}(|x|^3)$, valid for $|x| \leq \frac{1}{2}$:

$$\log(f_n(x)) = n\left(\frac{x}{\sqrt{n}} - \frac{(x/\sqrt{n})^2}{2} + \mathcal{O}\left(\left(\frac{x}{\sqrt{n}}\right)^3\right)\right) - \sqrt{n}x = -\frac{x^2}{2} + \mathcal{O}\left(\left(\frac{x}{\sqrt{n}}\right)^3\right).$$

As $n \to \infty$ the limit is $-\frac{x^2}{2}$, so

$$f_n(x) \xrightarrow{n\to\infty} e^{-\frac{x^2}{2}}.$$

To complete the demonstration we have to show that passing the limit through the integral sign is allowed.

We will use the dominated convergence Theorem:
Consider the function $g(x)$ defined as:

$$g(x) := \begin{cases} e^{-\frac{x^2}{2}}, & \text{if } x < 0 \\ (1+x)e^{-x}, & \text{if } x \geq 0. \end{cases}$$

This function is positive and integrable on $\mathbb{R}$.

We will prove that $0 \leq f_n(x) \leq g(x)$ for all n and x, it's obvious for $x \leq -\sqrt{n}$ since $f_n(x) = 0$.

To prove the inequality for $x \geq -\sqrt{n}$, take the logarithms:

$$\log(f_n(x)) = n\log\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x$$

and

$$\log(g(x)) = \begin{cases} -\frac{x^2}{2}, & \text{if } x < 0 \\ \log(1+x) - x, & \text{if } x \geq 0. \end{cases}$$

We have to look at the two cases separately:

If $-\sqrt{n} < x \leq 0$ then the difference $\log(f_n(x)) - \log(g(x))$ is:

$$\log(f_n(x)) + \frac{x^2}{2} = n \log\left(1 + \frac{x}{\sqrt{n}}\right) - \sqrt{n}x + \frac{x^2}{2}$$

which has first derivative:

$$\frac{n}{1 + \frac{x}{\sqrt{n}}} \frac{1}{\sqrt{n}} - \sqrt{n} + x = \frac{n}{\sqrt{n} + x} - \sqrt{n} + x = \frac{n + (x - \sqrt{n})(x + \sqrt{n})}{x + \sqrt{n}} = \frac{x^2}{x + \sqrt{n}}$$

positive for $-\sqrt{n} < x < 0$.

Therefore the function $\log(f_n(x)) - \log(g(x))$ increases for $-\sqrt{n} < x \leq 0$ but $\log(f_n(0)) - \log(g(0)) = 0$ so the function has to be negative in this interval.

If $x \geq 0$ then $\log(g(x)) = \log(1 + x) - x = f_1(x)$ and the inequality is trivial for $n = 1$.

For $n > 1$ consider this time the difference $\log(g(x)) - \log(f_n(x))$:

$$\log(1 + x) - x - \log(f_n(x)) = \log(1 + x) - x - n \log\left(1 + \frac{x}{\sqrt{n}}\right) + \sqrt{n}x.$$

This function has first derivative:

$$\frac{1}{1 + x} - 1 - \frac{n}{1 + \frac{x}{\sqrt{n}}} \frac{1}{\sqrt{n}} + \sqrt{n} = \frac{1 - 1 - x}{1 + x} + \frac{\sqrt{n}x}{\sqrt{n} + x} = \frac{(\sqrt{n} - 1)x^2}{(x + 1)(x + \sqrt{n})}$$

which is positive for $x > 0$ and $n \geq 2$.

Therefore the function $\log(g(x)) - \log(f_n(x))$ is increasing for $x \geq 0$ and $n \geq 2$, but once again $\log(f_n(0)) - \log(g(0)) = 0$ so the function is positive for $x > 0$.

To conclude, Lebesgue's dominated convergence Theorem assures that:

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) dx = \int_{-\infty}^{\infty} \lim_{n \to \infty} f_n(x) dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}.$$

□