POSITIVE INCREMENT

Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{2}$$

for $s \neq 0, \pm 1, \pm 2, \cdots$.

Proof. Start by noticing that:

$$\frac{1}{\Gamma(1-s)\Gamma(s)} = \frac{1}{-s\Gamma(-s)\Gamma(s)} \tag{3}$$

and remember Euler's infinite Product Formula:

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}}$$

proof of this relation can be found on our site.

Therefore, equation 3 can be rewritten as:

$$\frac{1}{-s\Gamma(-s)\Gamma(s)} = \frac{-s^2}{-s}\prod_{n=1}^{\infty} \frac{\left(1-\frac{s}{n}\right)\left(1+\frac{s}{n}\right)}{\left(1+\frac{1}{n}\right)^{-s}\left(1+\frac{1}{n}\right)^{s}} = s\prod_{n=1}^{\infty} 1 - \frac{s^2}{n^2} = \frac{\sin(\pi s)}{\pi} \tag{4}$$

where in the last equality we used the Euler Product expansion of the Sine Function:

$$\sin(s) = s\prod_{n=1}^{\infty} 1 - \frac{s^2}{\pi^2 n^2} \Rightarrow \sin(\pi s) = \pi s\prod_{n=1}^{\infty} 1 - \frac{\pi^2 s^2}{\pi^2 n^2} \Rightarrow \frac{\sin(\pi s)}{\pi} = s\prod_{n=1}^{\infty} 1 - \frac{s^2}{n^2}$$

□

Corollary 1.

$$\Gamma(s)\Gamma(-s) = -\frac{\pi}{s\sin(\pi s)} \tag{5}$$

and

$$\Gamma(1+s)\Gamma(1-s) = \frac{\pi s}{\sin(\pi s)} \tag{6}$$

for $s \neq 0, \pm 1, \pm 2, \cdots$.

Proof. To prove 5 use the fact that:

$$\Gamma(1+s) = s\Gamma(s)$$
$$\Downarrow$$
$$\Gamma(1-s) = -s\Gamma(-s)$$

in equation 2:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
$$\Downarrow$$
$$\Gamma(s)(-s\Gamma(-s)) = \frac{\pi}{\sin(\pi s)}$$
$$\Downarrow$$
$$\Gamma(s)\Gamma(-s) = -\frac{\pi}{s\sin(\pi s)}$$

While for 6, simply multiply both sides of 2 by s:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
$$\Downarrow$$
$$s\Gamma(s)\Gamma(1-s) = \frac{\pi s}{\sin(\pi s)}$$
$$\Downarrow$$
$$\Gamma(s+1)\Gamma(1-s) = \frac{\pi s}{\sin(\pi s)}.$$

□