POSITIVE INCREMENT

Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\int_x^{x+1} \frac{\Gamma'(s)}{\Gamma(s)} ds = \log x \tag{2}$$

for $Re(x) > 0$.

Proof. Compute this integral using the fact that:

$$\frac{\Gamma'(s)}{\Gamma(s)} = \frac{d}{ds}\left[\log \Gamma(s)\right]$$

therefore:

$$\begin{aligned} &\int_x^{x+1} \frac{\Gamma'(s)}{\Gamma(s)} ds = \int_x^{x+1} \frac{d}{ds}\left[\log\Gamma(s)\right] ds = \left|\log\Gamma(s)\right|_x^{x+1} \\ &= \log\Gamma(x+1) - \log\Gamma(x) = \log(x\Gamma(x)) - \log\Gamma(x) \\ &= \log x + \log\Gamma(x) - \log\Gamma(x) = \log x \end{aligned} \tag{3}$$

where we used the known property of the Gamma Function $\Gamma(x+1) = x\Gamma(x)$. □

Corollary 1.

$$\int_1^{n+1} \frac{\Gamma'(s)}{\Gamma(s)} ds = \log(n!) \tag{4}$$

for $n \in \mathbb{Z}_{\geq 0}$

Proof. Using the same reasoning we used for Theorem 1:

$$\int_1^{n+1} \frac{\Gamma'(s)}{\Gamma(s)} ds = \log \Gamma(n+1) - \log \Gamma(1) = \log(n!) \tag{5}$$

where we used the known property of the Gamma Function $\Gamma(n+1) = n!$.

□