POSITIVE INCREMENT

Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2\pi^{\frac{1}{2}} 2^{-2s} \Gamma(2s) \tag{2}$$

for $2s \neq 0, -1, -2, \cdots$.

Proof. Remember the definition of the **Beta Function** and its famous relation to the Gamma Function:

$$B(z_1, z_2) = \int_0^1 t^{z_1 - 1}(1-t)^{z_2 - 1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)} \tag{3}$$

true for $z_1, z_2 \in \mathbb{C}$ with $Re(z_1), Re(z_2) > 0$.

Fixing $z_1 = z_2 = s$ yields:

$$B(s, s) = \frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \int_0^1 t^{s-1}(1-t)^{s-1} dt \tag{4}$$

changing variable to $t = \frac{1+x}{2}$ we find:

$$\begin{aligned} \frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} &= \int_0^1 t^{s-1}(1-t)^{s-1} dt = \int_{-1}^1 \left(\frac{1+x}{2}\right)^{s-1} \left(1 - \frac{1+x}{2}\right)^{s-1} \frac{dx}{2} \\ &= \frac{1}{2} \int_{-1}^1 \left(\frac{1+x}{2}\right)^{s-1} \left(\frac{1-x}{2}\right)^{s-1} dx = \frac{1}{2^{2s-1}} \int_{-1}^1 \left(1 - x^2\right)^{s-1} dx. \end{aligned} \tag{5}$$

Notice that the integrated function $\left(1-x^2\right)^{s-1}$ is even and therefore:

$$\int_{-1}^{1}\left(1-x^2\right)^{s-1}dx = 2\int_{0}^{1}\left(1-x^2\right)^{s-1}dx.$$

Hence, equation 5 implies:

$$2^{2s-1}\Gamma(s)\Gamma(s) = 2\Gamma(2s)\int_{0}^{1}\left(1-x^2\right)^{s-1}dx. \tag{6}$$

Going back to the definition of the Beta Function 3 and substituting $t = x^2$, we also find:

$$B(z_1, z_2) = \int_0^1 2x \cdot x^{2z_1-2}(1-x^2)^{z_2-1}dx$$

fixing $z_1 = \frac{1}{2}$ and $z_2 = s$, this implies:

$$B\left(\frac{1}{2}, s\right) = 2\int_0^1 x \cdot x^{1-2}(1-x^2)^{s-1}dx = 2\int_0^1 (1-x^2)^{s-1}dx. \tag{7}$$

To conclude, combine equations 6 and 7:

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)B\left(\frac{1}{2}, s\right)$$

$$\Downarrow$$

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(s)}{\Gamma\left(\frac{1}{2}+s\right)}$$

$$\Downarrow$$

$$\Gamma(s)\Gamma\left(\frac{1}{2}+s\right) = 2^{1-2s}\Gamma(2s)\Gamma\left(\frac{1}{2}\right)$$

$$\Downarrow$$

$$\Gamma(s)\Gamma\left(\frac{1}{2}+s\right) = 2\pi^{\frac{1}{2}}2^{-2s}\Gamma(2s)$$

due to the fact that $\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}}$. □