Theorem 1.

$$\gamma = \int_0^1 \frac{1 - e^{-t} - e^{-\frac{1}{t}}}{t} \mathrm{d}t \tag{1}$$

where $\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right)$ *is the* ***Euler-Mascheroni constant.***

Proof. To prove this notice that:

$$\sum_{k=1}^{n} \frac{1}{k} = \int_0^1 1 + x + x^2 + \cdots + x^{n-1} \mathrm{d}x = \int_0^1 \frac{1 - x^n}{1 - x} \mathrm{d}x$$

changing variable to $1 - \frac{t}{n} = x$ we have:

$$\sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \frac{1 - x^n}{1 - x} \mathrm{d}x = \int_n^0 \frac{1 - \left(1 - \frac{t}{n}\right)^n}{1 - \left(1 - \frac{t}{n}\right)} \frac{-\mathrm{d}t}{n} = \int_0^n \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t.$$

While on the other hand:

$$\log n = \int_1^n \frac{1}{t} \mathrm{d}t.$$

Therefore:

$$\begin{aligned}
\gamma &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = \lim_{n\to\infty} \left(\int_0^n \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t - \int_1^n \frac{1}{t} \mathrm{d}t \right) \\
&= \lim_{n\to\infty} \left(\int_0^1 \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t + \int_1^n \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t - \int_1^n \frac{1}{t} \mathrm{d}t \right) \\
&= \lim_{n\to\infty} \left(\int_0^1 \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t + \int_1^n \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} - \frac{1}{t} \mathrm{d}t \right) \\
&= \lim_{n\to\infty} \left(\int_0^1 \frac{1 - \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t - \int_1^n \frac{\left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t \right)
\end{aligned} \tag{2}$$

$$
\begin{aligned}
&= \left(\int_0^1 \frac{1 - \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t - \lim_{n\to\infty} \int_1^n \frac{\left(1 - \frac{t}{n}\right)^n}{t} \mathrm{d}t \right) \\
&= \int_0^1 \frac{1 - e^{-t}}{t} \mathrm{d}t - \int_1^\infty \frac{e^{-t}}{t} \mathrm{d}t = \int_0^1 \frac{1 - e^{-t}}{t} \mathrm{d}t - \int_1^0 e^{-\frac{1}{t}} t \left(\frac{-\mathrm{d}t}{t^2} \right) \qquad (3) \\
&= \int_0^1 \frac{1 - e^{-t} - e^{-\frac{1}{t}}}{t} \mathrm{d}t.
\end{aligned}
$$

Were we switched the sings of integration and limit thanks to Lebesgue's dominated convergence Theorem, since for $0 \le t \le n$ we have $0 \le 1 - \left(1 - \frac{t}{n}\right)^n \le t$ by Bernoulli's inequality and $\left(1 - \frac{t}{n}\right)^n \le e^{-t}$ (because $1 - t \le e^{-t}$).

□