Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}} \right) \mathrm{d}t. \tag{2}$$

Remark 1. *The following is a more detailed version of the proof that can be found in [1].*

Proof. Start from the integral representation of the Euler-Mascheroni constant:

$$\gamma = \int_0^1 \frac{1 - e^{-t} - e^{-\frac{1}{t}}}{t} \mathrm{d}t. \tag{3}$$

Notice that:

$$\begin{aligned}
\gamma &= \int_0^1 \frac{1 - e^{-t}}{t} \mathrm{d}t - \int_0^1 \frac{e^{-\frac{1}{t}}}{t} \mathrm{d}t = \int_0^1 \frac{1 - e^{-t}}{t} \mathrm{d}t - \int_\infty^1 e^{-t} t \left(-\frac{\mathrm{d}t}{t^2} \right) \\
&= \int_0^1 \frac{1 - e^{-t}}{t} \mathrm{d}t - \int_1^\infty \frac{e^{-t}}{t} \mathrm{d}t = \int_0^1 \frac{1 - e^{-t}}{t} \mathrm{d}t - \int_0^\infty \frac{e^{-t}}{t} \mathrm{d}t + \int_0^1 \frac{e^{-t}}{t} \mathrm{d}t \\
&= \int_0^1 \frac{1}{t} \mathrm{d}t - \int_0^\infty \frac{e^{-t}}{t} \mathrm{d}t = \lim_{\delta \to 0} \left(\int_\delta^1 \frac{\mathrm{d}t}{t} - \int_\delta^\infty \frac{e^{-t}}{t} \mathrm{d}t \right).
\end{aligned} \tag{4}$$

Call now $\Delta = 1 - e^{\delta}$, since:

$$\int_\Delta^\delta \frac{\mathrm{d}t}{t} = \log\left(\frac{\delta}{\Delta}\right) = \log\left(\frac{\delta}{1 - e^{-\delta}}\right) \to 0 \quad \text{as } \delta \to 0.$$

We have:

$$\begin{aligned}\gamma &= \lim_{\delta\to 0}\left(\int_\delta^1 \frac{\mathrm{d}t}{t} - \int_\delta^\infty \frac{e^{-t}}{t}\mathrm{d}t\right) = \lim_{\delta\to 0}\left(\int_\Delta^\delta \frac{\mathrm{d}t}{t} + \int_\Delta^1 \frac{\mathrm{d}t}{t} - \int_\delta^\infty \frac{e^{-t}}{t}\mathrm{d}t\right)\\ &= \lim_{\delta\to 0}\left(\int_\Delta^1 \frac{\mathrm{d}t}{t} - \int_\delta^\infty \frac{e^{-t}}{t}\mathrm{d}t\right).\end{aligned} \tag{5}$$

Changing variable to $t = 1 - e^{-t}$ in the first integral we have:

$$\gamma = \lim_{\delta\to 0}\left(\int_\delta^\infty \frac{e^{-t}}{1-e^{-t}}\mathrm{d}t - \int_\delta^\infty \frac{e^{-t}}{t}\mathrm{d}t\right) = \int_0^\infty \frac{e^{-t}}{1-e^{-t}} - \frac{e^{-t}}{t}\mathrm{d}t. \tag{6}$$

Now notice that using the Weierstrass Product for the Gamma Function:

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^\infty \left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

$$\Downarrow$$

$$\frac{1}{s\Gamma(s)} = e^{\gamma s}\prod_{n=1}^\infty \left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

$$\Downarrow$$

$$\frac{1}{\Gamma(s+1)} = e^{\gamma s}\prod_{n=1}^\infty \left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}.$$

$$\Downarrow$$

$$\Gamma(s+1) = e^{-\gamma s}\prod_{n=1}^\infty \left(1+\frac{s}{n}\right)^{-1}e^{\frac{s}{n}}.$$

We can use this to compute the logarithmic derivative:

$$\psi(s+1) = \frac{\mathrm{d}\log\Gamma(s+1)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left(-\gamma s - \sum_{n=1}^\infty \log\left(1+\frac{s}{n}\right) + \frac{s}{n}\right) = -\gamma + \sum_{n=1}^\infty \frac{s}{n(n+s)}.$$

But $\log\Gamma(s+1) = \log s\Gamma(s) = \log s + \log\Gamma(s)$, therefore:

$$\frac{\mathrm{d}\log\Gamma(s+1)}{\mathrm{d}s} = \frac{1}{s} + \frac{\mathrm{d}\log\Gamma(s)}{\mathrm{d}s}$$

$$\Downarrow$$

$$\psi(s) = \frac{\mathrm{d}\log\Gamma(s)}{\mathrm{d}s} = -\gamma - \frac{1}{s} + \sum_{n=1}^\infty \frac{s}{n(n+s)} = -\gamma - \frac{1}{s} + \lim_{m\to\infty}\sum_{n=1}^m \left(\frac{1}{n} - \frac{1}{n+s}\right). \tag{7}$$

Notice that:

$$\frac{1}{s+n} = \int_0^\infty e^{-t(s+n)}\mathrm{d}t$$

when $n = 0, 1, 2, \cdots$ if $\mathrm{Re}(s) > 0$. Hence, using dominated convergence to switch the signs of sum and integration, we can write equation 7 as:

$$\begin{aligned}
\psi(s) &= -\gamma - \int_0^\infty e^{-st} + \lim_{m\to\infty} \int_0^\infty \sum_{n=1}^{m} \left(e^{-nt} - e^{-(n+s)t}\right) \mathrm{d}t \\
&= -\gamma + \lim_{m\to\infty} \int_0^\infty \sum_{n=1}^{m} \left(e^{-nt} - e^{-(n+s)t} - e^{-st}\right) \frac{1-e^{-t}}{1-e^{-t}} \mathrm{d}t \\
&= -\gamma + \lim_{m\to\infty} \int_0^\infty \sum_{n=1}^{m} \frac{e^{-nt} - e^{-(n+1)t} - e^{-(n+s)t} + e^{-(n+s+1)t} - e^{-st} + e^{-(s+1)t}}{1-e^{-t}} \mathrm{d}t.
\end{aligned} \tag{8}$$

This last sum is telescopic and can be calculated to yield:

$$\psi(s) = -\gamma + \lim_{m\to\infty} \int_0^\infty \frac{e^{-t} - e^{-st} - e^{-(m+1)t} + e^{-(m+s+1)t}}{1-e^{-t}} \mathrm{d}t \tag{9}$$

Substitute now γ using equation 6:

$$\begin{aligned}
\psi(s) &= \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}} \mathrm{d}t + \lim_{m\to\infty} \int_0^\infty \frac{e^{-t} - e^{-st} - e^{-(m+1)t} + e^{-(m+s+1)t}}{1-e^{-t}} \mathrm{d}t \\
&= \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}} \mathrm{d}t - \lim_{m\to\infty} \int_0^\infty \frac{1-e^{-st}}{1-e^{-t}} e^{-(m+1)t} \mathrm{d}t.
\end{aligned} \tag{10}$$

When $0 < t \leq 1$, the function $\left|\frac{1-e^{-st}}{1-e^{-t}}\right|$ is bounded with t and its limit as $t \to 0$ is finite. When $t \geq 1$:

$$\left|\frac{1-e^{-st}}{1-e^{-t}}\right| < \frac{1+\left|e^{-st}\right|}{1-e^{-t}} < \frac{2}{1-e^{-t}}.$$

Therefore we can find a number K independent of t such that, for $t \in [0, +\infty]$ we have:

$$\left|\frac{1-e^{-st}}{1-e^{-t}}\right| < K.$$

So we find:

$$\int_0^\infty \frac{1-e^{-st}}{1-e^{-t}} e^{-(n+1)t} \mathrm{d}t < K \int_0^\infty e^{-(n+1)t} \mathrm{d}t = \frac{K}{n+1} \to 0 \quad \text{as } n \to \infty.$$

This brings us to the conclusion:

$$\psi(s) = \int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-st}}{1-e^{-t}} \mathrm{d}t.$$

□

References

[1] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. 4th ed. Cambridge Mathematical Library. Cambridge University Press, 1996.