Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\Gamma(s) = -\frac{1}{2i\sin(\pi s)} \int_C (-t)^{s-1} e^{-t} dt \tag{2}$$

where the integration contour C *is a loop around the positive real axis; it starts at* $+\infty$*, encircles the origin once in the positive direction without enclosing any of the points* $t = \pm\pi i, \pm 3\pi i, \cdots$*, and returns to* $+\infty$*. The function* t^{-s} *has its principal value where* t *crosses the negative real axis and is continuous.*

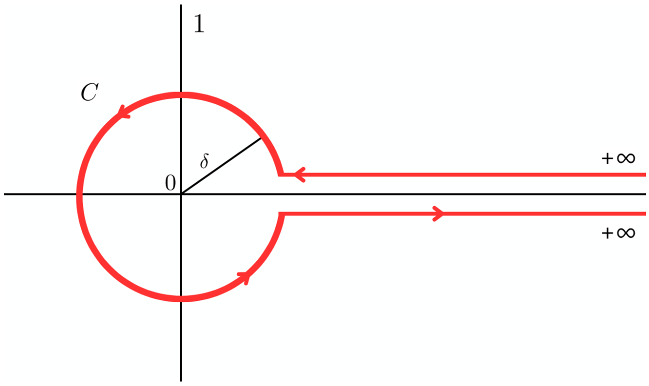

Figure 1: The Contour C

While this is a known property of the Gamma Function, its demonstrations often lack detail. Meanwhile, precise proof can be found in [1]; what we do here is explain clearly every passage.

Proof. Start by considering a contour, denoted as D. This contour starts just above a fixed real number ρ, moves parallel to the real axis towards a circle C_δ with a radius of $\delta > 0$, then travels counterclockwise around the origin along the circumference of the circle. Finally, it returns to a point just below ρ.

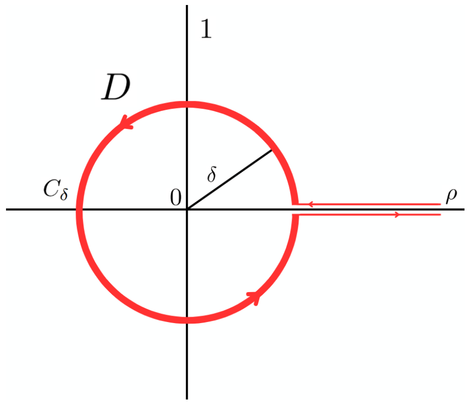

Figure 2: The Contour D

Remark 1. *Be mindful that the actual distance between the real axis and the contour is irrelevant as long as it is greater than zero; this is a consequence of Cauchy's Theorem. In the following steps, consider such distance* ≈ 0.

We will integrate the function $(-t)^{s-1}e^{-t}dt$ for $Re(s) > 0$ and $s \notin \mathbb{Z}$, the function $(-t)^{s-1}$ is well defined using the convention that $(-t)^{s-1} = e^{(s-1)\log(-t)}$ and $\log(-t)$ is purely real when t is on the negative part of the real axis.

We are therefore computing:

$$\int_D (-t)^{s-1}e^{-t}dt = \int_\rho^\delta (-t)^{s-1}e^{-t}dt + \int_{C_\delta} (-t)^{s-1}e^{-t}dt + \int_\delta^\rho (-t)^{s-1}e^{-t}dt.$$

On the first integral we have $\arg(-t) = -\pi \Rightarrow (-t)^{s-1} = e^{-i\pi(s-1)}t^{s-1}$ (remember that $\log t$ is purely real).

On the last integral $\arg(-t) = \pi \Rightarrow (-t)^{s-1} = e^{i\pi(s-1)}t^{s-1}$.

Meanwhile on the circle we can write $-t = \delta e^{i\theta}$.

Hence:

$$\int_D (-t)^{s-1} e^{-t} dt = \int_\rho^\delta e^{-i\pi(s-1)} t^{s-1} e^{-t} dt + \int_{-\pi}^{\pi} (\delta e^{i\theta})^{s-1} e^{\delta e^{i\theta}} \delta e^{i\theta} i d\theta + \int_\delta^\rho e^{i\pi(s-1)} t^{s-1} e^{-t} dt$$
$$= -e^{-i\pi(s-1)} \int_\delta^\rho t^{s-1} e^{-t} dt + e^{i\pi(s-1)} \int_\delta^\rho t^{s-1} e^{-t} dt + i\delta^s \int_{-\pi}^{\pi} e^{is\theta} e^{\delta(\cos(\theta)+i\sin(\theta))} d\theta$$
$$= -e^{-i\pi(s-1)} \int_\delta^\rho t^{s-1} e^{-t} dt + e^{i\pi(s-1)} \int_\delta^\rho t^{s-1} e^{-t} dt + i\delta^s \int_{-\pi}^{\pi} e^{is\theta} e^{\delta(\cos(\theta)+i\sin(\theta))} d\theta$$
$$= \left[e^{i\pi(s-1)} - e^{-i\pi(s-1)}\right] \int_\delta^\rho t^{s-1} e^{-t} dt + i\delta^s \int_{-\pi}^{\pi} e^{is\theta+\delta(\cos(\theta)+i\sin(\theta))} d\theta$$
$$= 2i\sin(\pi(s-1)) \int_\delta^\rho t^{s-1} e^{-t} dt + i\delta^s \int_{-\pi}^{\pi} e^{is\theta+\delta(\cos(\theta)+i\sin(\theta))} d\theta$$
$$= -2i\sin(\pi s) \int_\delta^\rho t^{s-1} e^{-t} dt + i\delta^s \int_{-\pi}^{\pi} e^{is\theta+\delta(\cos(\theta)+i\sin(\theta))} d\theta. \tag{3}$$

This is true for all positive values of $\delta \leq \rho$, therefore for $\delta \to 0$ we have, passing the limit under the integral sign thanks to Lebesgue's Theorem:

$$\int_{-\pi}^{\pi} e^{is\theta+\delta(\cos(\theta)+i\sin(\theta))} d\theta \to \int_{-\pi}^{\pi} e^{is\theta} d\theta$$
$$\Downarrow$$
$$i\delta^s \int_{-\pi}^{\pi} e^{is\theta+\delta(\cos(\theta)+i\sin(\theta))} d\theta \to 0$$
$$\Downarrow$$
$$\int_D (-t)^{s-1} e^{-t} dt = -2i\sin(\pi s) \int_0^\rho t^{s-1} e^{-t} dt.$$

However, this is true for all $\rho > 0$, making $\rho \to \infty$ and noticing that the contour C defined in the statement of Theorem 1 is the limit of the contour D as ρ approaches infinity, we have:

$$\int_C (-t)^{s-1} e^{-t} dt = -2i\sin(\pi s) \int_0^\infty t^{s-1} e^{-t} dt = -2i\sin(\pi s)\Gamma(s)$$
$$\Downarrow$$
$$\Gamma(s) = -\frac{1}{2i\sin(\pi s)} \int_C (-t)^{s-1} e^{-t} dt$$

□

Corollary 1.

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{-\infty}^{(0,+)} e^t t^{-s} dt \tag{4}$$

where the integration contour is C', a loop around the negative real axis; it starts at $-\infty$, encircles the origin once in the positive direction without enclosing any of the points $t = \pm\pi i, \pm 3\pi i, \cdots$, and returns to $-\infty$.
t^{-s} has its principal value where t crosses the positive real axis and is continuous.

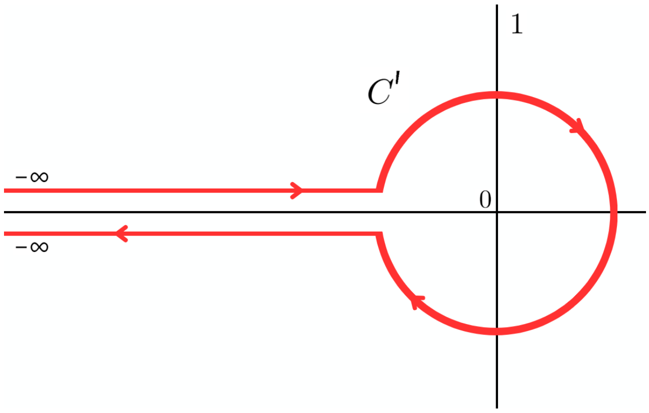

Figure 3: The Contour C'

Proof. Start by writing the result of Theorem 1 with $1-s$ instead of s:

$$\Gamma(1-s) = -\frac{1}{2i\sin(\pi(1-s))}\int_C (-t)^{1-s-1}e^{-t}dt$$
$$\Downarrow$$
$$\Gamma(1-s) = -\frac{1}{2i\sin(\pi s)}\int_C (-t)^{-s}e^{-t}dt$$
$$\Downarrow$$
$$\Gamma(1-s)\sin(\pi s) = -\frac{1}{2i}\int_C (-t)^{-s}e^{-t}dt.$$

Using now the formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
$$\Downarrow$$
$$\Gamma(1-s)\sin(\pi s) = \frac{\pi}{\Gamma(s)}$$

we have:

$$\frac{\pi}{\Gamma(s)} = -\frac{1}{2i}\int_C (-t)^{-s}e^{-t}dt$$
$$\Downarrow$$
$$\frac{1}{\Gamma(s)} = -\frac{1}{2\pi i}\int_C (-t)^{-s}e^{-t}dt.$$

Change now variable from $-t$ to t, notice that this changes the contour C defined in the main Theorem to the contour C' that appears in the Corollary.

Remark 2. *This change of variable, places the cut needed to have a well defined function* $\log(-t)$ *on the negative real axis.*

This change of variable yields:

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_{C'} e^t t^{-s}dt.$$

□

References

[1] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. 4th ed. Cambridge Mathematical Library. Cambridge University Press, 1996.