Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\Gamma(ns) = (2\pi)^{\frac{(1-n)}{2}} n^{ns-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(s + \frac{k}{n}\right) \tag{2}$$

for $ns \neq 0, -1, -2, \cdots$.

Proof. One of the main properties of the Gamma Function is the fact that $\Gamma(s+1) = s\Gamma(s)$, which implies:

$$\Gamma\left(s + \frac{k}{n}\right) = \left(s + \frac{k}{n} - 1\right)\Gamma\left(s + \frac{k}{n} - 1\right).$$

While Gauss's Expression of the Gamma Function is:

$$\Gamma(s) = \lim_{m\to\infty} \frac{m! m^s}{s(s+1)(s+2)\cdots(s+m)}. \tag{3}$$

Combining these two equations we obtain:

$$\begin{aligned} \Gamma\left(s + \frac{k}{n}\right) &= \left(s + \frac{k}{n} - 1\right)\Gamma\left(s + \frac{k}{n} - 1\right) \\ &= \lim_{m\to\infty} \left(s + \frac{k}{n} - 1\right) \frac{m! m^{s+\frac{k}{n}-1}}{\left(s + \frac{k}{n} - 1\right)\left(s + \frac{k}{n}\right)\cdots\left(s + \frac{k}{n} - 1 + m\right)}. \end{aligned} \tag{4}$$

Remember Stirling's approximation Formula for $m!$:

$$m! \approx \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$$

using this relation we can simplify equation 4:

$$\begin{aligned}
\Gamma\left(s+\frac{k}{n}\right) &= \lim_{m\to\infty}\left(s+\frac{k}{n}-1\right)\frac{m!m^{s+\frac{k}{n}-1}}{\left(s+\frac{k}{n}-1\right)\left(s+\frac{k}{n}\right)\cdots\left(s+\frac{k}{n}-1+m\right)} \\
&= \lim_{m\to\infty}\frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m m^{s+\frac{k}{n}-1}}{\left(s+\frac{k}{n}\right)\left(s+\frac{k}{n}+1\right)\cdots\left(s+\frac{k}{n}-1+m\right)} \\
&= \lim_{m\to\infty}\frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m m^{s+\frac{k}{n}-1}}{\left(s+\frac{k}{n}\right)\left(s+\frac{k}{n}+1\right)\cdots\left(s+\frac{k}{n}-1+m\right)}\cdot\frac{n^m}{n^m} \\
&= \lim_{m\to\infty}\frac{\sqrt{2\pi}\left(\frac{mn}{e}\right)^m m^{s+\frac{k}{n}-\frac{1}{2}}}{(ns+k)(ns+k+n)\cdots(ns+k-n+mn)}.
\end{aligned} \tag{5}$$

Consider now the product from $k=0$ to $n-1$:

$$\begin{aligned}
\prod_{k=0}^{n-1}\Gamma\left(s+\frac{k}{n}\right) &= \lim_{m\to\infty}(\sqrt{2\pi})^n\left(\frac{mn}{e}\right)^{nm}\prod_{k=0}^{n-1}\frac{m^{s+\frac{k}{n}-\frac{1}{2}}}{(ns+k)(ns+k+n)\cdots(ns+k-n+mn)} \\
&= \lim_{m\to\infty}(\sqrt{2\pi})^n\left(\frac{mn}{e}\right)^{nm}\frac{m^{ns-\frac{n}{2}+\sum_{k=0}^{n-1}\frac{k}{n}}}{(ns)(ns+1)\cdots(ns+n-1-n+mn)} \\
&= \lim_{m\to\infty}(\sqrt{2\pi})^n\left(\frac{mn}{e}\right)^{nm}\frac{m^{ns-\frac{n}{2}+\frac{1}{n}\cdot\frac{(n-1)n}{2}}}{(ns)(ns+1)\cdots(ns+n-1-n+mn)} \\
&= \lim_{m\to\infty}(\sqrt{2\pi})^n\left(\frac{mn}{e}\right)^{nm}\frac{m^{ns-\frac{n}{2}+\frac{n}{2}-\frac{1}{2}}}{(ns)(ns+1)\cdots(ns-1+mn)}.
\end{aligned} \tag{6}$$

Substitute mn with m, this doesn't change the fact that $m\to\infty$:

$$\begin{aligned}
\prod_{k=0}^{n-1}\Gamma\left(s+\frac{k}{n}\right) &= \lim_{m\to\infty}(\sqrt{2\pi})^n\left(\frac{mn}{e}\right)^{nm}\frac{m^{ns-\frac{1}{2}}}{(ns)(ns+1)\cdots(ns-1+mn)} \\
&= \lim_{m\to\infty}(\sqrt{2\pi})^n\left(\frac{m}{e}\right)^{m}\frac{m^{ns-\frac{1}{2}}n^{\frac{1}{2}-ns}}{(ns)(ns+1)\cdots(ns-1+m)} \\
&\quad\text{using again the fact that } m!\approx\sqrt{2\pi m}\left(\frac{m}{e}\right)^m: \\
&= \lim_{m\to\infty}(\sqrt{2\pi})^{n-1}\frac{m!m^{ns-1}n^{\frac{1}{2}-ns}}{(ns)(ns+1)\cdots(ns-1+m)} \\
&= \lim_{m\to\infty}(\sqrt{2\pi})^{n-1}\frac{m!m^{ns-1}n^{\frac{1}{2}-ns}}{(ns)(ns+1)\cdots(ns-1+m)}\cdot\frac{ns-1}{ns-1} \\
&= (\sqrt{2\pi})^{n-1}n^{\frac{1}{2}-ns}(ns-1)\lim_{m\to\infty}\frac{m!m^{ns-1}}{(ns-1)(ns)(ns+1)\cdots(ns-1+m)}.
\end{aligned} \tag{7}$$

The limit term is exactly in the form of equation 3 with $s=ns-1$, therefore:

$$\prod_{k=0}^{n-1}\Gamma\left(s+\frac{k}{n}\right)=(\sqrt{2\pi})^{n-1}n^{\frac{1}{2}-ns}(ns-1)\Gamma(ns-1)=(\sqrt{2\pi})^{n-1}n^{\frac{1}{2}-ns}\Gamma(ns)$$

$$\Downarrow$$

$$\Gamma(ns)=(2\pi)^{\frac{(1-n)}{2}}n^{ns-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(s+\frac{k}{n}\right).$$

$\square$