POSITIVE INCREMENT

Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!}{s(s+1)(s+2)\cdots(s+n)} n^s \tag{2}$$

true for $Re(s) > 0$. *This is sometimes referred to as "Gauss's expression" of the Gamma function.*

Proof. Remember the definition of e^t:

$$e^t = \lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n$$

$$\Downarrow$$

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n.$$

Use this equality in the definition of the Gamma Function to obtain:

$$\Gamma(s) = \int_0^\infty t^{s-1} \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n dt$$

the convergence of this integral is uniform, so we can switch the order of integration and limit, this yields:

$$\Gamma(s) = \lim_{n\to\infty} \int_0^\infty t^{s-1} \left(1 - \frac{t}{n}\right)^n dt = \lim_{n\to\infty} \int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt.$$

We proceed by integrating by parts:

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} \left| \frac{t^s}{s}\left(1-\frac{t}{n}\right)^n \right|_0^n - \int_0^n \frac{t^s}{s} n \left(1-\frac{t}{n}\right)^{n-1} \cdot \left(-\frac{1}{n}\right) dt \\ &= \lim_{n\to\infty} \frac{1}{s} \int_0^n t^s \left(1-\frac{t}{n}\right)^{n-1} dt.\end{aligned} \tag{3}$$

If we integrate by parts again, the computations are identical and we obtain:

$$\Gamma(s) = \lim_{n\to\infty} \frac{1}{s} \cdot \frac{n-1}{n(s+1)} \int_0^n t^{s+1} \left(1-\frac{t}{n}\right)^{n-2} dt$$

repeating this process n times yields:

$$\begin{aligned}\Gamma(s) &= \lim_{n\to\infty} \frac{1}{s} \cdot \frac{(n-1)!}{n^{n-1}(s+1)(s+2)\cdots(s+n-1)} \int_0^n t^{s+n-1} dt \\ &= \lim_{n\to\infty} \frac{1}{s} \cdot \frac{n(n-1)!}{nn^{n-1}(s+1)(s+2)\cdots(s+n-1)} \cdot \left| \frac{t^{s+n}}{s+n} \right|_0^n \\ &= \lim_{n\to\infty} \frac{1}{s} \cdot \frac{n!n^{s+n}}{n^n(s+1)(s+2)\cdots(s+n)} = \lim_{n\to\infty} \frac{n!}{s(s+1)(s+2)\cdots(s+n)} n^s.\end{aligned} \tag{4}$$

□