Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1} \tag{2}$$

Proof. Remember the Formula for the Gamma Function:

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!}{s(s+1)(s+2)\cdots(s+n)} n^s \tag{3}$$

proof of this formula can be found on our site.

Using the fact that $n = \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1}\right)$ we can rewrite it as:

$$\Gamma(s) = \lim_{n\to\infty} \frac{1}{s} \prod_{k=1}^{n} \frac{k}{s+k} n^s = \lim_{n\to\infty} \frac{1}{s} \prod_{k=1}^{n} \frac{k}{s+k} \left(\frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n}{n-1}\right)^s \tag{4}$$

now, using the fact that $\frac{k}{k-1} = 1 + \frac{1}{k-1}$ for any integer $k \geq 2$:

$$\begin{aligned}
\Gamma(s) &= \lim_{n\to\infty} \frac{1}{s} \prod_{k=1}^{n} \frac{k}{s+k} \left(\left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{n-1}\right)\right)^s \\
&= \lim_{n\to\infty} \frac{1}{s} \prod_{k=1}^{n} \frac{1}{\frac{s+k}{k}} \left(\left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{n-1}\right)\right)^s \\
&= \lim_{n\to\infty} \frac{1}{s} \prod_{k=1}^{n} \frac{1}{1 + \frac{s}{k}} \left(\left(1 + \frac{1}{1}\right) \cdot \left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{n-1}\right)\right)^s = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1}.
\end{aligned} \tag{5}$$

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