POSITIVE INCREMENT

Theorem 1. *The integral*

$$\int_0^\infty e^{-t}t^{s-1}dt$$

converges when $Re(s) > 0$*.*

Definition 1. *The Function defined for* $Re(s) > 0$ *as:*

$$\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt \tag{1}$$

is called the Gamma Function.

Proof. Fix $N > 1$ and break the integral into three parts:

$$\int_0^\infty e^{-t}t^{s-1}dt = \int_0^1 e^{-t}t^{s-1}dt + \int_1^N e^{-t}t^{s-1}dt + \int_N^\infty e^{-t}t^{s-1}dt$$

The first integral can be simply estimated, using the fact that when $t \in [0;1]$ we have $e^{-t} \leq 1$; therefore:

$$\left|\int_0^1 e^{-t}t^{s-1}dt\right| \leq \int_0^1 |t^{s-1}|dt = \frac{1}{s}$$

The middle integral also converges, as we can see that:

$$\left|\int_1^N e^{-t}t^{s-1}dt\right| \leq \int_1^N \left|e^{-t}t^{s-1}\right| dt \leq N^{s-1}\int_1^N \left|e^{-t}\right| dt \leq N^s$$

To see the convergence of the last integral, notice first that, whenever $Re(s) > 0$, we can find a big enough N such that:

$$t^{s-1} < e^{\frac{t}{2}}$$

whenever $t \geq N$.

Proving this fact is simply a matter of computing:

$$\lim_{t\to\infty} \frac{t^{s-1}}{e^{\frac{t}{2}}} = 0$$

this was done using De l'Hôpital's rule.

Hence, supposing that N is sufficiently big we can estimate the last term as:

$$\left|\int_N^\infty e^{-t}t^{s-1}dt\right| \leq \int_N^\infty \left|e^{-t}t^{s-1}\right| dt \leq \int_N^\infty \left|e^{\frac{t}{2}}e^{-t}\right| dt = \int_N^\infty \left|e^{\frac{-t}{2}}\right| dt = \frac{2}{e^{\frac{n}{2}}}$$

We have therefore proven that the whole integral converges. □

Theorem 2.

$$\Gamma(s+1) = s\Gamma(s)$$

for $Re(s) > 0$*.*

Proof. Integrating the definition by parts we obtain:

$$\Gamma(s+1) = \int_0^\infty e^{-t}t^s dt = \left|-e^{-t}t^s\right|_0^\infty + s\int_0^\infty e^{-t}t^{s-1} = s\Gamma(s).$$

□

Corollary 1.

$$\Gamma(n+1) = n! \tag{2}$$

for all positive integers n*.*

Proof. This is obviously implied by the theorem above, plus the fact that:

$$\Gamma(1) = \int_0^\infty e^{-t}t^0 = \int_0^\infty e^{-t} = 1$$

□

Corollary 2. *The Gamma function can be defined over the whole complex plane, as a meromorphic function with simple poles at the negative integers and zero.*

Proof. Theorem 2 implies:

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

$$\Downarrow$$

$$\Gamma(s) = \frac{\Gamma(s+2)}{s(s+1)}$$

$$\Downarrow$$

$$\Gamma(s) = \frac{\Gamma(s+n)}{s(s+1)(s+2)\cdots(s+n-1)}$$

for any positive integer n.

By definition $\Gamma(s+n)$ is analytic for $Re(s) > -n$ so the function on the right is meromorphic for $Re(s) > -n$ and has simple poles at $0, -1, -2, \cdots$.
The number n is arbitrary, therefore this equation extends the function to the whole complex plane.

□