Definition 1. *The Gamma Function is defined as:*

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1} dt \tag{1}$$

for $Re(s) > 0$.

Theorem 1.

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log(2\pi) + \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-ts}}{t}\mathrm{d}t \tag{2}$$

when $Re(s) > 0$.

Remark 1. *This proof comes in part from [1], we simply add some missing details.*

Proof. Start by considering the fact that:

$$\frac{\Gamma'(s)}{\Gamma(s)} = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{1 - e^{-t}}\right)\mathrm{d}t$$

Remark 2. *On our site you can find proof of this Integral representation.*

This implies:

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-st}e^{-t}}{1 - e^{-t}}\right)\mathrm{d}t = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{t}}{e^{t}} \cdot \frac{e^{-st}e^{-t}}{1 - e^{-t}}\right)\mathrm{d}t$$

$$\Downarrow$$

$$\frac{\Gamma'(s+1)}{\Gamma(s+1)} = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{e^{t} - 1}\right)\mathrm{d}t.$$

Using the fact that:

$$\log s = \int_0^{\infty} \frac{e^{-t} - e^{-st}}{t}\mathrm{d}t$$

and

$$\frac{1}{s} = \int_0^\infty e^{-st}\mathrm{d}t$$

we have:

$$\begin{aligned}
\frac{\Gamma'(s+1)}{\Gamma(s+1)} &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-st}}{e^t-1}\right)\mathrm{d}t = \int_0^\infty \left(\frac{e^{-t}-e^{-st}+e^{-st}}{t} - \frac{e^{-st}}{e^t-1}\right)\mathrm{d}t \\
&= \int_0^\infty \frac{e^{-t}-e^{-st}}{t}\mathrm{d}t + \int_0^\infty \left(\frac{e^{-st}}{t} - \frac{e^{-st}}{e^t-1}\right)\mathrm{d}t \\
&= \log s + \int_0^\infty \left(\frac{e^{-st}}{t} - \frac{e^{-st}}{e^t-1} + \frac{e^{-st}}{2} - \frac{e^{-st}}{2}\right)\mathrm{d}t \\
&= \log s + \int_0^\infty \frac{e^{-st}}{2}\mathrm{d}t + \int_0^\infty \left(\frac{e^{-st}}{t} - \frac{e^{-st}}{e^t-1} - \frac{e^{-st}}{2}\right)\mathrm{d}t
\end{aligned} \tag{3}$$

$$\Downarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\Gamma(s+1) = \frac{\Gamma'(s+1)}{\Gamma(s)} = \log s + \frac{1}{2s} - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)e^{-st}\mathrm{d}t.$$

The integrand in the last term is continuous as $t \to 0$, the function is also bounded as $t \to \infty$. Therefore, the integral converges uniformly when the real part of s is positive. Hence we can obtain $\log\Gamma(z+1)$ by integrating both sided from 1 to s, switching the two integrals in the last term:

$$\log\Gamma(s+1) = \int_1^s \log s\mathrm{d}s + \int_1^s \frac{1}{2s}\mathrm{d}s - \int_1^s \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)e^{-st}\mathrm{d}t\mathrm{d}s$$

$$\Downarrow$$

$$\log\Gamma(s+1) = s\log s - s + 1 + \frac{\log s}{2} - \int_0^\infty \int_1^s \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)e^{-st}\mathrm{d}s\mathrm{d}t$$

$$\Downarrow$$

$$\log\Gamma(s+1) = \left(s+\frac{1}{2}\right)\log s - s + 1 - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-st}-e^{-t}}{t}\mathrm{d}t.$$

Using the known property of the Gamma Function $s\Gamma(s) = \Gamma(s+1)$, we have $\log\Gamma(s+1) = \log s\Gamma(s) = \log s + \log\Gamma(s)$. Therefore:

$$\log s + \log\Gamma(s) = \left(s+\frac{1}{2}\right)\log s - s + 1 - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-st}-e^{-t}}{t}\mathrm{d}t$$

$$\Downarrow$$

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + 1 + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-st}}{t}\mathrm{d}t - \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-t}}{t}\mathrm{d}t. \tag{4}$$

We will evaluate the second of these integrals:

Define:

$$I := \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-t}}{t}\mathrm{d}t$$

and

$$J := \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t.$$

With these definitions, evaluating equation 4 for $s = \frac{1}{2}$ yields:

$$\log \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} + J - I$$
$$\Downarrow$$
$$\frac{1}{2}\log \pi = \frac{1}{2} + J - I.$$

Also, scaling the variable $t \to \frac{t}{2}$:

$$I = \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{\frac{t}{2}} - 1}\right) \frac{2e^{-\frac{t}{2}}}{t} \frac{\mathrm{d}t}{2} = \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{\frac{t}{2}} - 1}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t.$$

Therefore, we have:

$$\begin{aligned}
J - I &= \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t - \int_0^\infty \left(\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{\frac{t}{2}} - 1}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} - \frac{1}{2} + \frac{2}{t} - \frac{1}{e^{\frac{t}{2}} - 1}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{1}{t} + \frac{e^{\frac{t}{2}} - 1 - e^t + 1}{(e^t - 1)\left(e^{\frac{t}{2}} - 1\right)}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t = \int_0^\infty \left(\frac{1}{t} + \frac{e^{\frac{t}{2}}\left(1 - e^{\frac{t}{2}}\right)}{(e^t - 1)\left(e^{\frac{t}{2}} - 1\right)}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{1}{t} - \frac{e^{\frac{t}{2}}}{e^t - 1}\right) \frac{e^{-\frac{t}{2}}}{t} \mathrm{d}t = \int_0^\infty \left(\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t - 1}\right) \frac{1}{t} \mathrm{d}t.
\end{aligned} \tag{5}$$

And so

$$\begin{aligned}
J = (J - I) + I &= \int_0^\infty \left(\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t - 1}\right) \frac{1}{t} \mathrm{d}t + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-t}}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{e^{-\frac{t}{2}}}{t} - \frac{1}{e^t - 1} + \frac{e^{-t}}{2} - \frac{e^{-t}}{t} + \frac{e^{-t}}{e^t - 1}\right) \frac{1}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{e^{-\frac{t}{2}} - e^{-t}}{t} + \frac{e^{-t}}{2} + \frac{e^{-t} - 1}{e^t - 1}\right) \frac{1}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{e^{-\frac{t}{2}} - e^{-t}}{t} + \frac{e^{-t}}{2} + \frac{e^{-t}}{e^{-t}} \cdot \frac{e^{-t} - 1}{e^t - 1}\right) \frac{1}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{e^{-\frac{t}{2}} - e^{-t}}{t} + \frac{e^{-t}}{2} + e^{-t}\left(\frac{e^{-t} - 1}{1 - e^{-t}}\right)\right) \frac{1}{t} \mathrm{d}t \\
&= \int_0^\infty \left(\frac{e^{-\frac{t}{2}} - e^{-t}}{t} + \frac{e^{-t}}{2} - e^{-t}\right) \frac{1}{t} \mathrm{d}t = \int_0^\infty \left(\frac{e^{-\frac{t}{2}} - e^{-t}}{t} - \frac{e^{-t}}{2}\right) \frac{1}{t} \mathrm{d}t.
\end{aligned} \tag{6}$$

Now notice that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right)=\frac{(t+2)e^{-\frac{t}{2}}-2(t+1)e^{-t}}{2t^2}=\frac{e^{-\frac{t}{2}}}{2t}+\frac{e^{-\frac{t}{2}}}{t^2}-\frac{e^{-t}}{t}-\frac{e^{-t}}{t^2}$$

$$\Downarrow$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right)=\left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right)\frac{1}{t}+\frac{e^{-\frac{t}{2}}}{2t}-\frac{e^{-t}}{t}$$

$$\Downarrow$$

$$\left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right)\frac{1}{t}=-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right)-\frac{e^{-\frac{t}{2}}}{2t}+\frac{e^{-t}}{t}.$$

Therefore:

$$\begin{aligned} J &= \int_0^\infty \left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}-\frac{e^{-t}}{2}\right)\frac{1}{t}\mathrm{d}t \\ &= \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right)-\frac{e^{-\frac{t}{2}}}{2t}+\frac{e^{-t}}{t}-\frac{e^{-t}}{2t}\mathrm{d}t \\ &= -\left|\frac{e^{-\frac{t}{2}}-e^{-t}}{t}\right|_0^\infty-\int_0^\infty \frac{e^{-\frac{t}{2}}}{2t}+\frac{2e^{-t}}{2t}-\frac{e^{-t}}{2t}\mathrm{d}t \\ &= \frac{1}{2}+\frac{1}{2}\int_0^\infty \frac{e^{-t}-e^{-\frac{t}{2}}}{t}\mathrm{d}t=\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}. \end{aligned} \tag{7}$$

So

$$I=\frac{1}{2}+J-\frac{1}{2}\log\pi=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\log\frac{1}{2}-\frac{1}{2}\log\pi=1-\frac{1}{2}\log 2\pi.$$

In conclusion, we have Binet's Formula:

$$\log\Gamma(s)=\left(s-\frac{1}{2}\right)\log s-s+\frac{1}{2}\log 2\pi+\int_0^\infty\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-st}}{t}\mathrm{d}t. \tag{8}$$

□

References

[1] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. 4th ed. Cambridge Mathematical Library. Cambridge University Press, 1996.